LECTURE I

Hamiltonian and inverse-Hamiltonian dynamical systems

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Hamiltonian and inverse-Hamiltonian vector fields

- $M-$ manifold,
- $C(M)$ $-$ the space of all smooth real-valued functions on M,
- duality map: $T^*M \times TM \to \mathbb{R}$, $\langle \gamma, v \rangle (x) = v(\gamma)(x)$, $v \in TM$, $\gamma \in T^*M$,
- tensor fields of second order:
- $$(\Pi^{ij}) = \Pi, \qquad \Pi(\gamma_1, \gamma_2) \in \mathbb{R}, \quad \Pi\gamma \in TM,$$
- $$(\Omega_{ij}) = \Omega, \qquad \Omega(v_1, v_2) \in \mathbb{R}, \quad \Omega v \in T^*M,$$
- $$(\Phi^i_j) = \Phi, \qquad \Phi(v, \gamma) \in \mathbb{R}, \quad \Phi v \in TM, \ \Phi\gamma \in T^*M.$$

Hamiltonian and inverse-Hamiltonian vector fields

Definition

Poisson tensor Π of co-rank r on M of $\dim M = m$, is a bivector, which in local coordinates $(x_1, ..., x_m)$ takes the form

$$\Pi = \sum_{i<j}^{m} \Pi^{ij} \partial_i \wedge \partial_j, \qquad \partial_k \equiv \frac{\partial}{\partial x_k}, \tag{1.1}$$

where

$$\sum_r \left(\Pi^{jr} \partial_r \Pi^{ik} + \Pi^{ir} \partial_r \Pi^{kj} + \Pi^{kr} \partial_r \Pi^{ji} \right) = 0.$$

- The kernel of Π is spanned by r exact one-forms

$$\ker \Pi = Sp\{dc(x)\}_{i=1,\dots,r}. \tag{1.2}$$

- A function $f \in C(M)$ is called a Casimir function of Π if $df \in \ker \Pi$.

Definition

A vector field X_H, related to function $H \in C(M)$ by

$$X_H = \Pi dH \tag{1.3}$$

is called the Hamiltonian vector field with respect to Π. H is called a Hamiltonian function.

Definition

A presymplectic tensor Ω of co-rank r on M is a two-form that is closed, i.e. $d\Omega = 0$. In a local coordinate system on M

$$\Omega = \sum_{i<j}^{m} \Omega_{ij}\ dx_i \wedge dx_j, \tag{1.4}$$

where the closeness condition is

$$\partial_i \Omega_{jk} + \partial_k \Omega_{ij} + \partial_j \Omega_{ki} = 0.$$

Hamiltonian and inverse-Hamiltonian vector fields

- The kernel of Ω is an integrable distribution

$$\ker \Omega = Sp\{Z_i\}_{i=1,\ldots,r}, \qquad [Z_i, Z_j] = 0.$$

Definition

A vector field X^H, related to function $H \in C(M)$ by

$$\Omega X^H = dH, \tag{1.5}$$

is called the inverse-Hamiltonian vector field with respect to Ω.

Non-degenerate case. Symplectic manifold

- If co-rank $r = 0$, then $m = 2n$.
- Non-degenerated closed two-form ω: symplectic form.
- Non-degenerated Poisson bivector π: implectic bivector.
- In that case
$$\omega^{-1} = \pi \quad \Longrightarrow \quad X_H = X^H.$$
- Implectic tensor π induces a Lie algebra on the space $C(M)$:
- $$\{.,.\}_\pi : \; C(M) \times C(M) \to C(M).$$
- $$\{F, G\}_\pi(x) := \langle dF, \pi dG \rangle (x) = \pi(x)(dF, dG). \tag{1.6}$$

- In fact, there holds
- $$\{F, G\}_\pi = -\{G, F\}_\pi \qquad \text{antisymmetry}$$
- $$\{F, GH\}_\pi = \{F, G\}_\pi H + G\{F, H\}_\pi \qquad \text{Leibniz rule}$$
- $$\{F, \{G, H\}_\pi\}_\pi + \ c.p. = 0 \qquad \text{Jacobi indentity}$$
- This particular Lie algebra is called a Poisson algebra.

Non-degenerate case. Symplectic manifold

- The Poisson bracket (1.6) can be defined by symplectic form $\omega = \pi^{-1}$, as:
-

$$\{F, G\}^{\omega} := \omega(X_F, X_G) = \langle \omega X_F, X_G \rangle = \left\langle \omega X^F, X_G \right\rangle = \langle dF, \pi dG \rangle$$
$$= \pi(dF, dG) = \{F, G\}_{\Pi} \qquad (1.7)$$

- According to Darboux theorem, there always exist local coordinates $(x_1, \ldots, x_{2n})$ (global on $M = \mathbb{R}^{2n}$), such that

$$\pi = \sum_{i=1}^{n} \partial_{x_i} \wedge \partial_{x_{n+i}}, \qquad (\pi^{ij}) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

$$\omega = \sum_{i=1}^{n} dx_{n+i} \wedge dx_i, \qquad (\omega_{ij}) = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

- $\omega\pi = \pi\omega = I_{2n}, \quad (x_1, \ldots, x_{2n})-$ canonical coordinates.

Dynamical systems on M. Flows.

- $$u_t = X_H(u) = \pi dH(u) = \pi\omega X^H(u) = X^H(u). \qquad (1.8)$$
- Let Q be a position manifold (configuration space) of dim $Q = n$. Then, a Poisson manifold is $M = T^*Q$ of dim $M = 2n$, with local chart (global for $Q = \mathbb{R}^n$) $(q, p)-$ canonical position and momenta coordinates. M is called a phase space.

Hamiltonian flows on pseudo-Riemann space

- Let (Q, g) be a pseudo-Riemann space with covariant metric tensor g and local coordinates $(q^1, ..., q^n)$. Let $G = g^{-1}$ be a contravariant metric tensor.
- The Levi-Civita connection components are defined by

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{r=1}^{n} G^{ir}(\partial_k g_{rj} + \partial_j g_{kr} - \partial_r g_{jk}), \qquad \partial_k \equiv \frac{\partial}{\partial q^k}.$$

- The equations

$$q^i_{tt} + \Gamma^i_{jk} q^j_t q^k_t = G^{ik}\partial_k V(q), \qquad i = 1, ..., n \tag{1.9}$$

  describe the motion of a particle in the pseudo-Riemann space Q with the metric g, under potential $V(q)$.
- Obviously, for $G = I$ (1.9) reduces to Newton equations of motion.
- Eqs. (1.9) are equivalent to Lagrangian equations

$$\delta L = 0, \qquad L = \frac{1}{2}\sum_{i,j} g_{ij} q^i_t q^j_t - V(q).$$

Hamiltonian flows on pseudo-Riemann space

- We can pass to the standard Hamiltonian description on $M = T^*Q$, where

$$h(q,p) = \sum_{i=1}^{n} q_t^i \frac{\partial L}{\partial q_t^i} - L = \frac{1}{2}\sum_{i,j} G^{ij} p_i p_j + V(q),$$

$$p_i = \frac{\partial L}{\partial q_t^i} = \sum_j g_{ij} q_t^j.$$

- The equations of motion on M are

$$\begin{bmatrix} q \\ p \end{bmatrix}_t = \pi dh = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial h}{\partial q} \\ \frac{\partial h}{\partial p} \end{bmatrix} = X_h. \qquad (1.10)$$

- For $V(q) = 0$ we have geodesic motion.

Henon-Heiles system

- On $M = T^*Q \ni (q_1, q_2, p_1, p_2)$ consider Hamiltonian function

$$h = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1^3 + \frac{1}{2}q_1 q_2^2$$

- and related Hamiltonian dynamics

$$\begin{aligned} q_{1t} &= p_1 \\ q_{2t} &= p_2 \\ p_{1t} &= -3q_1^2 - \frac{1}{2}q_2^2 \\ p_{2t} &= -q_1 q_2. \end{aligned}$$

Liouville integrable systems

- Consider

$$u_t = X_h(u) = \pi dh(u), \qquad u = (q,p)^T. \tag{1.11}$$

- Function $f(u)$ is called constant of motion of (1.11) if

$$\{f(u), h(u)\}_\pi = 0.$$

- Let M be a symplectic manifold of dim $M = 2n$, with dynamical system given by $h(u)$.
- Assume that the system has n constant of motion $f_1, \ldots, f_n$ in involution

$$\{f_i, f_j\}_\pi = 0$$

for all i, j and differentials df_i are independent at each $T_u^* M$.

Liouville integrable systems

- Fix a point $a \in \mathbb{R}^n$ and consider a map

$$F: \; M \to \mathbb{R}^n, \qquad F = (f_1, ..., f_n).$$

- It is regular map by the assumption abaut df_i, so $M_a := F^{-1}(a)$ is a smooth n-dimentional submanifold for each $a \in \mathbb{R}^n$.
- Moreover it is a Lagrangian submanifold as :

$$\omega(X_{f_i}, X_{f_j}) = (f_i, f_j\}_\pi = 0$$

and $X_{f_i} \subset T_u M_a$.

Theorem

(Liouville-Arnold)

1. If M_a is compact and connected, then it is diffeomorphic to $T^n = \mathbb{R}^n/\mathbb{Z}^n$, the $n-$dimensional torus.
2. In the vicinity of M_a there are cannonical coordinates $(I_1, ..., I_n, \varphi_1, ..., \varphi_n)$ $(0 \le \varphi_i \le 2\pi)$, action-angle coordinates, in which (1.11) reads
$$I_{it} = 0, \quad \varphi_{it} = \varsigma_i(I_1, ..., I_n) = \frac{\partial h(I_1, ..., I_n)}{\partial I_i}.$$

- It is quasi periodic motion on T^n. For the realization of periodic motion, additionally:
$$\sum_{i=1}^{n} n_i \varsigma_i = 0, \qquad n_i \in \mathbb{Z}.$$
- In principle, the system is integrable in quadratures. In practice, we can do it in very particular cases, only. For example, if we can find a coordinates in which h separates onto n one-dimensional systems.

Canonical transformations

- Consider: $(q,p) \to (Q,P)$

$$\pi = \sum_i \partial_{q_i} \wedge \partial_{p_i}, \quad \omega = \pi^{-1} = \sum_i dp_i \wedge dq_i$$

$$\Downarrow$$

$$\pi' = \sum_i \partial_{Q_i} \wedge \partial_{P_i}, \quad \omega' = \pi'^{-1} = \sum_i dP_i \wedge dQ_i.$$

- Consider smooth function $F(q,P)$, called generating function of canonical transformation, such that $\left|\frac{\partial^2 F}{\partial q \partial P}\right| \neq 0$.
- Define

$$p_i = \frac{\partial F}{\partial q_i} \equiv F_{q_i}, \quad Q_i = \frac{\partial F}{\partial P_i} \equiv F_{P_i}.$$

- Then,

$$\begin{aligned} dp &= F_{qq} dq + F_{Pq} dP \\ dQ &= F_{Pq} dq + F_{PP} dP \end{aligned} \quad \Longrightarrow \quad \begin{aligned} dq &= F_{Pq}^{-1} dQ - F_{Pq}^{-1} F_{PP} dP \\ dp &= F_{qq} F_{Pq}^{-1} dQ + \left(F_{qP} - F_{qq} F_{Pq}^{-1} F_{PP} \right) \end{aligned}$$

Canonical transformations

- so we have
$$\sum_i dp_i \wedge dq_i = \sum_i dP_i \wedge dQ_i.$$
- In a similar fasion one finds:
$$F(q,Q) \quad \Longrightarrow \quad p = F_q, \quad P = -F_Q,$$
$$F(p,P) \quad \Longrightarrow \quad q = -F_p, \quad Q = -F_P,$$
$$F(Q,p) \quad \Longrightarrow \quad q = F_p, \quad P = F_Q.$$